DET KGL. DANSKE VIDENSKABERNES SELSKAB
MATEMATISK-FYSISKE MEDDELELSER, BIND XXIII, NR. 11.

DEDICATED TO PROFESSOR NIELS BOHR ON THE OCCASION OF HIS 60TH BIRTHDAY

ON THE THEORY OF β-DECAY II

BY

STEFAN ROZENTAL

KØBENHAVN
I KOMMISSION HOS EJNAR MUNKSGAARD
1945

Printed in Denmark.
Bianco Lunos Bogtrykkeri A/S
REPRINTED 1948 TUTEIN OG KOCH

Introduction.

Among the experimental tests of a theory of nuclear forces one of the most important is the comparison of its results concerning the process of β-decay with experimental evidence. From the assumptions on which a theory of nuclear forces is based, it should be possible not only to derive the magnitude and the general character of these forces, but also to calculate the lifetime of β-radioactive elements and to predict the shape of the energy distribution curve of the β-rays emitted. A theory based on the assumption of a nuclear meson field must furthermore yield a value of the lifetime of a free meson, which agrees with the value of the lifetime of mesons in cosmic radiation.

Various types of a meson theory of nuclear forces have been developed, but, so far, none of them has proved to be satisfactory in all respects. Particularly, the scattering of fast neutrons by protons is a phenomenon for which the experimental results seem to be very difficult to bring into harmony with the theoretical expectation. One type of a meson theory has been put forward by Møller and Rosenfeld [1] who regard the meson field as a superposition of two components, one of which is described by a vector and the other by a pseudoscalar wave-function. Against this theory (in the following denoted as the MR-theory) the objection has been raised that the results concerning the character of the scattering of fast neutrons by protons are in disagreement with known experiments. In fact, it follows from the theory that a beam of fast neutrons will be scattered in such a way that the intensity in the direction anti-parallel to the incident beam is larger (about 1.5 times) than the intensity in the direction perpendicular to it, the angles being measured in a system of reference where the centre of gravity of the two colliding particles is at rest. The experiments, on the contrary, seem to indicate that the ratio of the intensities anti-parallel and perpendicular to the neutron beam is smaller than unity. Two remarks should be made in this connection. First, the accuracy of the measurements is so far not sufficient to allow a decisive statement. Even the experiments of CHAMPION and POWELL [2], who use the powerful photographic method, are not accurate enough. It is possible, however, that an improvement in the geometrical arrangement will lead to conclusive results. Second, as pointed out by HULTHÉN [3], the discrepancy in question, if real, will appear in every consistent meson theory of nuclear forces and not in the MR-theory, only.

Owing to the progress in the experimental technique during the last years, a very extensive and well-founded knowledge of the shape of different β-ray spectra could be gathered. In contrast to earlier investigations, the recent measurements seem to indicate [4] that, in the case of the so-called allowed transitions, the spectra coincide—at any rate for not too small energies of the electron emitted—with the curve given by the original formula of FERMI [5]. In the region of lower energies, a certain deviation from the Fermi law has been found. Whether the β-spectrum really differs from the pure Fermi distribution law and is given by another formula as, for example, the generalized Fermi law [13], which also follows from the present investigation, is difficult to decide with certainty. In the case of positron emitters, this latter law leads to a curve which differs from the pure Fermi law only little and in the same direction as the experimentally found spectra, and the elements examined were obviously all positron emitters. On the other hand, the deviation may be due to the scattering of the β-rays from the support, an effect which, if sufficiently large, also would explain the deviation from an original distribution given by the pure Fermi law. If it is true that the β-spectrum is given by a generalized Fermi formula, the deviation from the pure Fermi law would be more pronounced in the case of electron emitters which are not found among the light nuclei so far investigated. All these considerations show that it may be of some value to examine what kind of law for the β-decay follows from the MR-theory.

In an earlier paper ([6], in the following quoted as I), the theory of β-decay was developed for light nuclei from the point of view of the MR-theory of the nuclear meson field. The calculations resulted in a formula for the disintegration probability which besides terms of the Fermi type includes terms differing from them. Whether such terms really are of significance for the shape of the spectrum or not depends on the relative magnitude of the coefficients. Since the number of constants involved in these coefficients is very large, the discussion of all possibilities is rather troublesome, and the formula would become still more complicated in the case of heavier radioactive elements. It is possible, however, to reduce the number of independent constants and to determine their values, and the final formula for the energy distribution becomes easy to survey.

The notations used in the present paper are the same as in I.

Determination of the Constants.

The method applied to the derivation of the decay probability of heavy nuclei is similar to that used in I. The starting point is a Hamiltonian describing a system of heavy particles (nucleons), light particles (electrons, neutrinos), and the meson field (vector and pseudoscalar). With this expression for the Hamiltonian the probability is derived for a process in which a neutron is transformed into a proton at the same time as a neutrino in a negative energy state disappears and an electron in a positive energy state is created. This probability (per unit time) is given by the formula

$$P(E_s) = \frac{2\pi}{\hbar}\,\delta(W + E_\sigma - E_s)\,|(n, s\,|\,H_\beta\,|\,n_0, \sigma)\,|^2, \tag{1}$$

where H_β is the part of the Hamiltonian responsible for the β-emission, n_0 and n denote the initial and the final states of the nucleus, and σ and s the states of the neutrino and the electron, respectively. The energies of the electron and the neutrino are denoted by E_s and E_σ, while $W = E_{n_0} - E_n$ is the total energy released in the β-process. The expression H_β is built up of wave-functions and contains a number of universal constants. Thus, the constants g_1 and g_2 govern the strength of the coupling between the nucleons and the vector meson field, while the interaction between the nucleons and the pseudoscalar part of the meson field depends on the value of the constants f_1 and f_2. Similarly, the constants $\breve{g}_1, \breve{g}_2$ and $\breve{f}_1, \breve{f}_2$ appear in the terms which describe the interaction between the light particles and the vector and the pseudoscalar meson field, respectively. All these constants have the dimensions of electric charge.

Furthermore, the expression H_β contains four constants of a character deviating from that of the f's and g's. The Hamiltonian of the nuclear system is not defined in a unique way by the demand of relativistic invariance. In fact, we can write down four expressions (I, formula 10) which are relativistically invariant and which, therefore, can be added to the Hamiltonian provided with the constant factors η, η', η'', η''', respectively. According as such a coefficient is put equal to 1 or 0, the corresponding term will or will not appear in the Hamiltonian.

One of the main points of the MR-theory is in the expression for the nuclear force to make disappear terms with a singularity of dipole type. The singular terms originating from the vector and the pseudoscalar meson field, respectively, become equal with opposite sign, and cancel each other, if

$$f_2^2 = g_2^2 .$$

A more effective reduction in the number of constants involved can be achieved by following MØLLER [7], who has developed a formalism in which the vector and the pseudoscalar parts of the meson field are united into one five-dimensional scheme. The constants connected with the two kinds of fields are, then, no longer independent and have to satisfy the following relations:

$$\left.\begin{aligned} f_1 &= g_1 \\ f_2 &= -g_2 \\ \breve{f}_1 &= \breve{g}_1 \\ \breve{f}_2 &= -\breve{g}_2 . \end{aligned}\right\} \quad (2)$$

Moreover, if this formalism is adopted, it will be quite natural to demand that the four terms provided with the factors η, η', η'' and η''' and added to the Hamiltonian should be invariant with respect to the group of rotations in the whole five-dimensional space in question. From this assumption we get as a necessary condition

$$\left.\begin{aligned} \eta &= \eta'' \\ \eta' &= \eta''' . \end{aligned}\right\} \quad (3)$$

In this way, the number of constants is reduced from twelve to six. The values of the universal constants g_1, g_2, $\breve{g}_1$, $\breve{g}_2$ can now be fixed by using some experimentally known properties of atomic nuclei and mesons.

As regards the two constants g_1, they and g_2 are found from the value of the binding energy of the deuteron and the range of the nuclear force. We have approximately

$$\frac{g_1^2}{4\pi\hbar c} = \frac{1}{35}, \qquad \frac{g_2^2}{4\pi\hbar c} = \frac{1}{15}. \tag{4}$$

As regards the values of the constants $\breve{g}_1$ and $\breve{g}_2$ we have, as already mentioned, to consider the connection between the β-decay of light elements and the radioactive properties of cosmic ray mesons. In the meson theory the β-disintegration is imagined to take place in two steps. In the first step, a meson is virtually created under the transition of a nucleon from the neutron to the proton state. In the second step, the meson is annihilated into an electron and an antineutrino. The probability of the second step, for which the decay constant of the meson is a direct measure, is an essential part of the probability of the whole complex process of β-decay.

Thus, the constants $\breve{g}_1$ and $\breve{g}_2$ appear both in the expression found in I for the decay constant λ_{rad} of a light β-radioactive nucleus (the transition being an allowed one) and in the decay constant of a free meson. According as the meson is of vector or of pseudoscalar type, we get for the decay probability the following formulae [8, 9]:

$$\lambda_V = \frac{M_m c^2}{4\pi\hbar}\,\frac{1}{\hbar c}\left(\frac{2}{3}\,\breve{g}_1^2 + \frac{1}{3}\,\breve{g}_2^2\right) \tag{5a}$$

$$\lambda_{PS} = \frac{M_m c^2}{4\pi\hbar}\,\frac{1}{\hbar c}\left(\breve{g}_1 - \frac{\mu}{M_m}\,\breve{g}_2\right)^2 \tag{5b}$$

where M_m and μ are the masses of the meson and of the electron, respectively. No agreement between the lifetimes of light β-radioactive nuclei and the lifetime of a meson can be obtained, if we consider vector mesons, only. For every possible choice of $\breve{g}_1$ and $\breve{g}_2$ compatible with the empirical values of λ_{rad}, the calculated lifetime of the vector meson turns out to be about 1000 times smaller than the measured one which is of the order of magnitude of 2×10^{-6} sec. [10]. As soon as we assume, however, that the cosmic ray mesons are of the pseudoscalar type with the decay constant (5b), the discrepancy disappears [11], if only

$$\breve{g}_1^2 \ll \breve{g}_2^2. \tag{6}$$

The ratio between $\breve{g}_1$ and $\breve{g}_2$ has to be chosen of the order of magnitude of 0.02—0.05. From this assumption it follows that the vector mesons created simultaneously with the pseudoscalar mesons in the upper layers of the atmosphere will, due to their very short lifetime, disintegrate before they can reach the surface of the earth and, consequently, only pseudoscalar mesons will be found in our laboratories where the measurements are performed.

If the condition (6) is fulfilled, we get, from the disintegration formula in I, an energy distribution which, for $\eta = 1$ follows the curve given by the Fermi function

$$F(E) = E\sqrt{E^2 - 1}\,(W - E)^2 \tag{7}$$

and for $\eta = 0$ the curve

$$\Phi(E) = F(E)\left(1 \pm \frac{1}{E}\right), \tag{8}$$

where the upper sign refers to the emission of positrons and the lower to the emission of electrons. Actually, we have

$$P(E)\,dE = \frac{2}{\pi^3}\left(\frac{\mu}{M_m}\right)^4\left(\frac{\mu c^2}{\hbar}\right)F(E)\left(1 \pm \frac{(1-\eta)^2}{E}\right)\frac{g_2^2\,\breve{g}_2^2}{(\hbar c)^2}\,2\,|M|^2\,dE. \tag{9}$$

Here, $|M|^2$ is a matrix element depending on the wave-functions of the nucleons. It is independent of the value of η', but is different according as the constant η has the value 0 or 1. The empirical selection rules seem to indicate that the first of these matrix elements should be preferred, since it is of the proper spin-dependent type [12]. It will, therefore, be appropriate to put the constant η equal to 0.

It should be mentioned here that the spectrum given by the function (8) is the same as that which, as pointed out by FIERZ [13], follows from the most general Fermi theory. For positrons, the shape of the spectrum differs from the pure Fermi distribution (7) only in the region of lower energies (cf. I, Fig. 1), but it is difficult to say whether the deviation from the distribution (7) found experimentally (all light elements for which the transition is allowed are positron emitters) is of the type given by the generalized Fermi function (8).

Derivation of the Disintegration Formula.

The formula for the decay probability derived in I applies to elements with a small nuclear charge Z fulfilling the condition

$$Z\alpha \ll 1, \tag{10}$$

where α is the fine structure constant

$$\alpha = \frac{e^2}{\hbar c} = \frac{1}{137}.$$

The simplifications carried out in the preceding Section now allow to extend the calculation to heavier elements for which the condition (10) is not satisfied.

The method used in this case is similar to that adopted in I, where all details of the calculations can be found.

We start with the expression for the Hamiltonian and go on with the evaluation of the quantity (1). It must be noticed that, despite the relation (6), it is not allowed in the Hamiltonian to cancel all terms with $\breve{g}_1$. Although, in the final formula, the terms with the coefficient $\breve{g}_1$ or $\breve{g}_1^2$ generally are small compared with terms with $\breve{g}_2^2$, they may in some cases play a decisive part, when the other terms happen to vanish due to selection rules.

In order to find the matrix element in (1) belonging to the transition of the nucleus from the initial stage n_0 to the final state n, we insert the wave-functions of the light particles. The neutrino is not affected by the charge of the nucleus and can therefore be described by means af a plane wave while, for the electrons, exact solutions of the wave equation have to be applied.

The expressions involving such wave-functions are integrals extended over the volume of the nucleus. Since we may assume that the radial part of the electron wave-function $\varphi_s(x)$ (x stands for all spatial coordinates, and s denotes the electron state) does not vary appreciably inside the nucleus, we can replace the radial part of the wave-function by its value at the boundary of the nucleus, a value which in its turn is nearly equal to the value—taken in the same point—of an exact solution of the wave equation for a Coulomb field.

After the summation over all states of the neutrino belonging to the energy E_σ, we get the formula

$$P(E)\,dE = \frac{2\pi}{\hbar}\frac{1}{(hc)^3}(W-E)^2\,dE\sum_s U, \tag{11}$$

where the terms U are composed of the electron wave-functions and the Dirac spin operators. The summation is extended over all electron states with the energy E_s. The four-component wave-functions in the Coulomb field can be used in the shape given by Rose [14], who denotes the radial parts of the two first and the two last components by $f_\varkappa$ and $g_\varkappa$, respectively, where $\varkappa$ is a quantity connected with the total angular momentum number. The functions $f_\varkappa$ and $g_\varkappa$ are given by the formula

$$\left.\begin{aligned} \begin{Bmatrix} f_\varkappa \\ g_\varkappa \end{Bmatrix} &= \frac{\sqrt{1\mp E}\,(2pr)^{\gamma-1}\sqrt{p}\,e^{\pi\frac{\alpha ZE}{p}}\left|\Gamma\left(\gamma+i\frac{\alpha ZE}{p}\right)\right|}{\sqrt{\pi}\,\Gamma(2\gamma+1)}\times \\ &\times\left\{e^{-ipr}\sqrt{-\left(\left(\varkappa-i\frac{\alpha Z}{p}\right)\left(\gamma+i\frac{\alpha ZE}{p}\right)\right)}\cdot\right. \\ &\left. F\left(\gamma+1+i\frac{\alpha ZE}{p},2\gamma+1;2ipr\right)\mp c.\,c.\right\}. \end{aligned}\right\} \tag{12}$$

Here. p is the momentum of the electron, F is the confluent hypergeometric function, Γ is the gamma function, and

$$\gamma = \sqrt{\varkappa^2-\alpha^2 Z^2}.$$

All quantities are expressed in atomic units. With these notations the sum in (11) transforms into a sum $\sum_\varkappa V$ of terms, each of which is a function of the $f_\varkappa$, $g_\varkappa$, and the unit vector

$$\vec{n} = \frac{\vec{x}}{|\vec{x}|}.$$

Generally, it is unnecessary to extend the summation over all $\varkappa$. The functions $f_\varkappa$ and $g_\varkappa$ are taken at the boundary of the nucleus: $r = r_{\text{nucl}} \ll 1$. From (12) it is seen that the values of $f_\varkappa$ and $g_\varkappa$ for r_{nucl} decrease very rapidly with increasing $|\varkappa|$. In most cases, we can confine ourselves to the terms with $\varkappa = \pm 1$, but in some terms V, particularly in those which involve derivatives of the wave function, it is necessary to take also terms with $|\varkappa| = 2$ into consideration.

Discussion of the Decay Formula.

The final formula giving the probability $P(E)\,dE$ per unit time for the emission of an electron with the energy (expressed in units μc^2) between E and $E + dE$ consists of a number of terms. The most important terms are the "Fermi terms", *i. e.* terms which in the limit of light nuclei would give a pure or generalized Fermi distribution. Other terms containing an extra factor $\sqrt{E^2 - 1}$ or $(E^2 - 1)$ are in addition multiplied by $\frac{\mu}{M_m} \approx \frac{1}{200}$ or $\left(\frac{\mu}{M_m}\right)^2 \approx \frac{1}{40000}$ and may, therefore, be cancelled. The "Fermi terms" can be classified according as the constant factor is $\breve{g}_2^2$, $\breve{g}_1\breve{g}_2$ or $\breve{g}_1^2$. Mostly, only terms with the large coefficient $\breve{g}_2^2$ have to be retained, the others being small owing to the relation (6). One of such terms with $\breve{g}_2^2$ is

$$\frac{g_2^2\,\breve{g}_2^2}{(\hbar c)^2}\, 4\, \omega_2\,(B, B)\,(f_{+1})^2. \tag{13}$$

Here, $\omega_2\,(B, B)$ (cf. I, formula 42) is a matrix element of the Gamow-Teller type [12]:

$$\omega_2\,(B, B) = \iint \vec{B}\,(x')\,\vec{B}^*(x)\,dx'\,dx\,, \tag{14}$$

where

$$\vec{B} = \sum_{i=1}^{N} \int \psi_n^*\, Q^{(i)}\, \vec{\sigma}^{(i)}\, \psi_{n_0}\, dx^{(i)} \cdots dx^{(i-1)}\, dx^{(i+1)} \cdots dx^{(N)}$$

is an integral over the wave-functions Ψ of the nucleus in the initial and final states, $Q^{(i)}$ is an operator transforming the i'th nucleon from the neutron state into a proton state, N is the number of the nucleons in the nucleus, and $\vec{\sigma}^{(i)}$ is the Pauli spin operator to the i'th nucleon.

The most important case is that of an allowed transition, *i. e.* a process for which the matrix element (14) attains its maximum value

$$\omega_2(BB) = \iint (B_x^* B_x + B_y^* B_y + B_z^* B_z)\, dx'\, dx \approx 3.$$

In this case, the result is quite independent of the value of η'. All terms with $\breve{g}_1^2$ and $\breve{g}_1\breve{g}_2$ are small and can be cancelled. The remaining terms are the same for $\eta' = 0$ and $\eta' = 1$. In the limit of small Z, *i. e.* when condition (10) is satisfied, and for that part of the β-spectrum where

$$\frac{\alpha ZE}{p} \ll 1,$$

we get, as it was to be expected, the formula (9) with $\eta = 0$. Actually, it is seen from (12), when expanding in series, that f_{-1} is, for $Z\alpha \ll 1$ small of the order of magnitude of $r_{\text{nucl}} \ll 1$ as compared with f_{+1}, and the other terms with g_2^2, *viz.*

$$\frac{g_2^2 \breve{g}_2}{(\hbar c)^2}\, 4\,(f_{-1})^2\, \omega_{10}(B, B), \tag{15}$$

where

$$\left.\begin{aligned} \omega_{10}(B, B) = \iint \Big\{ & (\vec{B}(x')\,\vec{B}^*(x))\,(\vec{n}(x)\cdot\vec{n}(x')) \\ & + (\vec{B}(x')\,\vec{n}(x'))\,(\vec{n}(x)\,\vec{B}^*(x)) \\ & + (\vec{B}(x')\,\vec{n}(x))\,(\vec{n}(x')\,\vec{B}^*(x)) \Big\}\, dx'\, dx \end{aligned}\right\} \tag{16}$$

are negligible, so that (13) is the only remaining term. We get a β-spectrum of a generalized Fermi type described by the function (8).

With increasing Z we get a deviation from the Fermi distribution and the energy distribution is now characterized by a function

$$F(E)\; C(E, Z)$$

which is found in substituting the proper function f_{+1} into (13). The correctional factor C becomes, of course, equal to 1 for small Z. Moreover, with increasing Z the function f_{-1} remains small as compared with f_{+1} only in the region where $\frac{\alpha Z}{p} E \ll 1$, *i. e.* for sufficiently great energies. This means that, if only ω_{10} does not vanish, the correction introduced by the term (15) is not negligible and makes itself perceptible in a part of the β-spectrum which lies below a certain electron energy, this limit energy becoming higher and higher with increasing nuclear charge Z.

In the case of a forbidden transition, *i. e.* when ω_2 (B, B) vanishes, the situation is changed. The term (13) disappears, and if ω_{10} is also equal to 0, other terms with the coefficient $\breve{g}_1 \breve{g}_2$ and with other matrix elements will now be responsible for the general character of the β-spectrum. In contrast to the case of allowed transitions, there is now a difference between the formulae for $\eta' = 0$ and $\eta' = 1$. In the disintegration formula we have terms of pure Fermi type and of the type $F(E)(E^2 - 1)$ (cf. I, Fig. 2). The decay constant, which is simply the integral over $P(E)$ from $E = 1$ to $E = W$, is now smaller and, consequently, the lifetime is longer than in the case of an allowed transition with the same maximum energy W, since the constant coefficient $\breve{g}_1 \breve{g}_2$ in $P(E)$ is smaller than the coefficient $\breve{g}_2^2$ appearing in the decay formula for allowed transitions. Also the selection rules for the transition are now given by the differing form of the matrix elements appearing in the disintegration formula.

Summary.

A theory of β-decay for elements with high nuclear charges is developed on the lines of the special meson theory proposed by Møller and Rosenfeld. The values of the universal constants involved have been determined from the requirement of a consistent qualitative description of the nuclear forces, the β-process and the disintegration of the meson. The discussion of the disintegration formula indicates that, for an allowed transition, the spectrum is represented by the generalized Fermi formula. For a forbidden transition also terms of other type can occur

The author wishes to express his deep gratitude to Professor Niels Bohr for his kind interest in this work. Professor Bohr's highly stimulating discussions and neverfailing help have been of the utmost significance to me during my stay at the Institute of Theoretical Physics, Copenhagen. The present investigation has been made possible through a stipend from the Rask-Ørsted Foundation to whom the writer is greatly indebted.

Due to war circumstances, part of the investigation had to be carried out in Sweden. The author cordially thanks Professor Oskar Klein for the excellent working conditions and the hospitality offered to him at the Institute of Mechanics and Mathematical Physics, University of Stockholm. Also to the Nobel Foundation, Stockholm, he is much obliged for a grant.

References.

1. C. Møller and L. Rosenfeld, D. Kgl. Danske Vidensk. Selskab, Mat.-fys. Medd. **XVII**, Nr. 8, 1940.
2. F. C. Champion and C. F. Powell, Proc. Roy. Soc. (A) **183**, 64, 1944.
3. L. Hulthén, Private communication.
4. Cf. for instance, K. Siegbahn, Arkiv f. Mat. Fys. Astr. **30 A**, No. 20, 1944.
5. E. Fermi, ZS. f. Phys. **88**, 161, 1934.
6. S. Rozental, D. Kgl. Danske Vidensk. Selskab, Mat.-fys. Medd. **XVIII**, Nr. 7, 1941.
7. C. Møller, D. Kgl. Danske Vidensk. Selskab, Mat.-fys. Medd. **XVIII**, Nr. 6, 1941.
8. H. Yukawa, S. Sakata, M. Kobayashi and M. Taketani, Proc. Phys.-Math. Soc. Japan **20**, 720, 1938.
9. T. S. Chang, D. Kgl. Danske Vidensk. Selskab, Mat.-fys. Medd. **XIX**, Nr. 10, 1942.
10. F. Rasetti, Phys. Rev. **60**, 198, 1941.
11. S. Rozental, Phys. Rev. **60**, 612, 1941.
12. G. Gamow and E. Teller, Phys. Rev. **49**, 895, 1936.
13. M. Fierz, ZS. f. Phys. **104**, 553, 1937.
14. M. E. Rose, Phys. Rev. **51**, 484, 1937.

Indleveret til Selskabet den 6. August 1945.
Færdig fra Trykkeriet den 26. September 1945.